Pascal Sum Formula

Note Title 11/27/2009

$$S_n^p := \sum_{i=1}^{n} i^p$$

then: $$\sum_{i=1}^{p} \binom{p+1}{i} S_n^{p+1-i} = (n+1)^{p+1} - 1$$

proof: examples - see below

$$\sum_{i=1}^{p+1} \underbrace{\binom{p+1}{i}}_{=\binom{p+1}{p+1-i}} S_n^{p+1-i} = \sum_{i=0}^{p} \binom{p+1}{i} S_n^{i} = \sum_{i=0}^{p} \binom{p+1}{i} \sum_{k=1}^{n} k^i =$$

$$= \sum_{k=1}^{n} \left(\sum_{i=0}^{p} \binom{p+1}{i} k^i + \underbrace{\binom{p+1}{p+1}}_{=1} k^{p+1} - \binom{p+1}{p+1} k^{p+1} \right) =$$

$$= \sum_{k=1}^{n} \underbrace{\sum_{i=0}^{p+1} \binom{p+1}{i} k^{i}} - \sum_{k=1}^{n} k^{p+1}$$

$$= \sum_{k=1}^{n} (k+1)^{p+1} - \sum_{k=1}^{n} k^{p+1} = \sum_{k=2}^{n+1} k^{p+1} - \sum_{k=1}^{n} k^{p+1} = (n+1)^{p+1} - 1$$

$\square$

Examples:

$$S_n^0 = (n+1)^{0+1} - 1 = n+1-1 = n = 1^0 + 2^0 + \dots + n^0$$

$$(1+1) S_n^1 + S_n^0 = (n+1)^{1+1} - 1$$

$$\iff 2 S_n^1 + n = (n+1)^2 - 1 = (n+1-1)(n+1-(-1)) = n(n+2)$$

$$\iff 2 S_n^1 = n(n+1) \iff S_n^1 = \frac{n(n+1)}{2} = 1+2+3+\dots+n$$

$$\underbrace{(2+1)}\, S_n^2 + \binom{3}{2} S_n^1 + \binom{3}{3} S_n^0 = (n+1)^3 - 1$$

$$\Leftrightarrow \quad 3 S_n^2 + 3 S_n^1 + S_n^0 = n^3 + 3n^2 + 3n + 1 - 1$$

$$\Leftrightarrow \quad 3 S_n^2 + \underbrace{3\,\frac{n(n+1)}{2}} + n = n^3 + 3n^2 + 3n$$

$$\Leftrightarrow \quad 3 S_n^2 + \frac{3}{2} n^2 + \frac{3}{2} n + n = n^3 + 3n^2 + 3n$$

$$\Leftrightarrow \quad S_n^2 = n^3 + \frac{3}{2} n^2 + \frac{1}{2} n$$

$$\Leftrightarrow \quad S_n^2 = \frac{n^3}{3} + \frac{1}{2} n^2 + \frac{1}{6} n = \frac{2n^3 + 3n^2 + n}{6} = \frac{n}{6}\left(2n^2 + n + 2n + 1\right)$$

$$= \frac{n}{6}\left(n(2n+1) + 2n + 1\right) = \frac{n}{6}(n+1)(2n+1) = 1^2 + 2^2 + \ldots + n^2$$