Minimum of a Generalized Reciprocal Sum

Problem:

$$\min \sum_i \frac{q_i}{x_i^\alpha} \quad \text{mit } x \in \mathbb{R}_+^n, \sum_i x_i = \beta \text{ und } 1 < \alpha < \infty$$

Hölder's Inequality

$a, b \in \mathbb{R}^n$; $1 < p < \infty$ $(1 < p' < \infty)$

$$\frac{1}{p} + \frac{1}{p'} = 1$$

$$\sum_i a_i b_i \leq \left(\sum_i a_i^p\right)^{1/p} \left(\sum_i b_i^{p'}\right)^{1/p'} \Leftrightarrow \frac{\sum_i a_i b_i}{\left(\sum_i a_i^p\right)^{1/p}} \leq \left(\sum_i b_i^{p'}\right)^{1/p'}$$

It would be perfect, if $\sum_i a_i^p$ was a constant and $\sum_i b_i^{p'}$ was $\sum_i \frac{q_i}{x_i^\alpha}$.

So let's define a and b to satisfy these requirements:

$a_i := x_i^{1/p}$, then $\sum_i a_i^p = \sum_i x_i = \beta$

and $b_i := \left(\frac{q_i}{x_i^\alpha}\right)^{1/p'}$, then $\sum_i b_i^{p'} = \sum_i \frac{q_i}{x_i^\alpha}$

What about the dot product?

$$\sum_i a_i b_i = \sum_i x_i^{1/p} \cdot \frac{q_i^{1/p'}}{x_i^{\alpha/p'}} = \sum_i x^{\frac{1}{p} - \frac{\alpha}{p'}} \cdot q_i^{\frac{1}{p'}}$$

The left side of the equation should be independent of x.

So we want to choose p and p' in a way to yield:

$$x^{\frac{1}{p}-\frac{\alpha}{p'}} = 1 \;\forall x>0 \Leftrightarrow \frac{1}{p} - \frac{\alpha}{p'} = 0 \Leftrightarrow \frac{1}{p} = \frac{\alpha}{p'}$$

With $\frac{1}{p} + \frac{1}{p'} = 1 \Leftrightarrow \frac{1}{p'} = 1 - \frac{1}{p} \Leftrightarrow p' = \frac{1}{1-\frac{1}{p}}$:

$$\frac{1}{p} = \frac{\alpha}{\frac{1}{1-\frac{1}{p}}} = \alpha\left(1 - \frac{1}{p}\right) \Leftrightarrow (1+\alpha)\frac{1}{p} = \alpha$$

$$\Leftrightarrow \frac{1+\alpha}{\alpha} = p$$

$$\Leftrightarrow p' = \frac{1}{1-\frac{\alpha}{1+\alpha}} = 1+\alpha$$

$$p := \frac{1+\alpha}{\alpha}, \quad p' = 1+\alpha$$

Thus we get: $a_i b_i = x_i^{\frac{\alpha}{1+\alpha}} \cdot \frac{q_i^{\frac{1}{1+\alpha}}}{x_i^{\frac{\alpha}{1+\alpha}}} = q_i^{\frac{1}{1+\alpha}}$

$$\Rightarrow \sum_i a_i b_i = \sum_i q_i^{\frac{1}{1+\alpha}}$$

$$\overset{\text{Hölder}}{\Longrightarrow} \quad \sum_i q_i^{\frac{1}{p'}} \leq \beta^{\frac{1}{p}} \cdot \left(\sum_i \frac{q_i}{x_i^{\alpha}}\right)^{\frac{1}{p'}}$$

$$\Leftrightarrow \frac{\left(\sum_i q_i^{\frac{1}{p'}}\right)^{p'}}{\beta^{\frac{p'}{p}}} \leq \sum_i \frac{q_i}{x_i^{\alpha}}$$

$$\frac{\left(\sum_i \sqrt[p']{q_i}\right)^{p'}}{\beta^{p'/p}} = \frac{\left(\sum_i \sqrt[\alpha+1]{q_i}\right)^{\alpha+1}}{\beta^{\alpha}}$$ is the minimum.

Let's now show how to reach this minimum using Hölder's inequality again.

Equality case of Hölder's inequality:

$$\sum_i a_k b_k = \left(\sum_i a_k^p\right)^{\frac{1}{p}} \cdot \left(\sum_i b_k^{p'}\right)^{\frac{1}{p'}} \iff \exists \lambda \in \mathbb{R} : \quad a_k^p = \lambda b_k^{p'}$$

$$\left(x_i^{\frac{1}{p}}\right)^p = \lambda \left(\frac{q_i}{x_i^\alpha}\right)^{\frac{1}{p'} \cdot p'} \iff x_i = \lambda \frac{q_i}{x_i^\alpha} \iff x_i^{1+\alpha} = \lambda q_i$$

$$\iff x_i^{p'} = \lambda q_i \iff x_i = (\lambda q_i)^{\frac{1}{p'}} = \lambda^{\frac{1}{p'}} q_i^{\frac{1}{p'}}$$

We insert this into $\beta = \sum_i x_i = \lambda^{\frac{1}{p'}} \sum_i q_i^{\frac{1}{p'}}$

$$\iff \lambda^{\frac{1}{p'}} = \beta \cdot \frac{1}{\sum_i \sqrt[p']{q_i}}$$

$$\Rightarrow x_i = \beta \cdot \frac{\sqrt[p']{q_i}}{\sum_i \sqrt[p']{q_i}} = \beta \cdot \frac{\sqrt[1+\alpha]{q_i}}{\sum_i \sqrt[1+\alpha]{q_i}}$$

To sum up:

The optimal x^* is:

$$x_i^* = \beta \cdot \frac{\sqrt[1+\alpha]{q_i}}{\sum_i \sqrt[1+\alpha]{q_i}}$$

with a minimum value of:

$$\frac{\left(\sum_i \sqrt[\alpha+1]{q_i}\right)^{\alpha+1}}{\beta^\alpha} = \sum_i \frac{q_i}{x_i^\alpha}$$

Let's verify the last result:

$$\sum_i \frac{q_i}{\left(\beta \cdot \frac{\sqrt[\alpha+1]{q_i}}{\sum_i \sqrt[\alpha+1]{q_i}}\right)^\alpha} = \frac{\left(\sum_i \sqrt[\alpha+1]{q_i}\right)^\alpha}{\beta^\alpha} \cdot \sum_i \underbrace{\frac{q_i}{\left(\sqrt[\alpha+1]{q_i}\right)^\alpha}}_{q_i^{1-\frac{\alpha}{\alpha+1}} = q_i^{\frac{1}{\alpha+1}}} =$$

$$= \frac{\left(\sum_i \sqrt[\alpha+1]{q_i}\right)^\alpha}{\beta^\alpha} \cdot \sum_i \sqrt[\alpha+1]{q_i} = \frac{\left(\sum_i \sqrt[\alpha+1]{q_i}\right)^{\alpha+1}}{\beta^\alpha}$$